Stability, Competitive Equilibrium, and Full Substitutability

John William Hatfield
McCombs School of Business
University of Texas at Austin

(Based on work with S.D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp)
In matching, auction theory, and exchange economies with indivisible goods, some form of “substitutability” is usually essential for guaranteeing the existence of equilibria (if we allow arbitrary “simple” preferences).

**Matching:** Kelso & Crawford (1982); Roth (1984); Hatfield & Milgrom (2005); Ostrovsky (2008); Hatfield & Kominers (2012). . .


And underpins a number of high-profile applications:

- **Spectrum auctions**  
  (Ausubel, 2006)

- **Oil and natural gas supply-chains**  
  (Ostrovsky, 2008)

- **“Swap” deals in exchange markets**  
  (Milgrom, 2009)

- **Natural gas auctions**  
  (Milgrom & Strulovici, 2009)

- **Securities auctions**  
  (Klemperer, 2010; Baldwin–Klemperer, 2014, 2015)
Defining Substitutability

But all of these papers use different definitions of substitutability:
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- Is substitutability defined over *objects* or *contracts*?
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- How does the definition deal with indifferences?
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We show that all of these works use the same notion of substitutability.
## Overview of Results

**Matching:**

<table>
<thead>
<tr>
<th>Author/Year</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roth (1984)</td>
<td>(S)</td>
</tr>
<tr>
<td>Ostrovsky (2008), Hatfield &amp; Kominers (2012)</td>
<td>(SSS)+(CSC)</td>
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<tr>
<td>Hatfield et al. (2013)</td>
<td>(FS)</td>
</tr>
</tbody>
</table>

In fully general trading networks:

1. choice-language full substitutability,
### Overview of Results

**Exchange:**

<table>
<thead>
<tr>
<th>Source</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gul &amp; Stachetti (1999)</td>
<td>(GS), (SI), (NC); (SM)</td>
</tr>
<tr>
<td>Baldwin &amp; Klemperer (2019)</td>
<td>(GS), (OS)</td>
</tr>
</tbody>
</table>

In fully general trading networks:

1. choice-language full substitutability,*
2. demand-language full substitutability,*
### Auction Theory:

<table>
<thead>
<tr>
<th>Study</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milgrom–Strulovici (2009)</td>
<td>(WS)/(SS); (SM)</td>
</tr>
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<td>Gul–Stachetti (2000)</td>
<td>(GS), (SI), (NC); (SM)</td>
</tr>
<tr>
<td>Milgrom (2000)</td>
<td>(SM)</td>
</tr>
<tr>
<td>Ausubel–Milgrom (2002)</td>
<td>(S); (SM)</td>
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<tr>
<td>Ausubel (2006)</td>
<td>(S)</td>
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<tr>
<td>Sun–Yang (2009)</td>
<td>(GSC), (GSI); (SM)</td>
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In fully general trading networks:

1. choice-language full substitutability,*
2. demand-language full substitutability,*
3. indicator-language full substitutability,*
4. submodularity of indirect utility,
5. object-language full substitutability,
6. the single improvement property,
7. the no-complementarities condition, and
Overview of Results

OR / Math:

Fujishige–Yang (2003) — (GS), \((M^{\mathcal{h}}-c)\)

Shioura–Tamura (2015) — (GS), \((M^{\mathcal{h}}-c)\), (SS); (SM); \((M^{\mathcal{h}}-EXC)\) (PRJ-GS), (SWGS), (GS+LAD)

Paes Leme (2014) — (GS), \((M^{\mathcal{h}}-c)\), …

Baldwin–Klemperer (2014) — (GS), \((D^{n}_{OS})\), (TH)

In fully general trading networks:

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8. \(M^{\mathcal{h}}\)-concavity of the valuation
Overview of Results

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**In fully general trading networks:**

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6. the single improvement property,
7. the no-complementarities condition, and
8. $M^\natural$-concavity of the valuation are all equivalent.
## Overview of Results

### Theorem (Transformations)

*Full substitutability is preserved under*

1. *trade endowments and obligations,*
2. *mergers,* and
3. *limited liability.*
Overview of Results

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<thead>
<tr>
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</tr>
</thead>
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</tbody>
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Overview of Results

Theorem (Transformations)

Full substitutability is preserved under

1. trade endowments and obligations,
2. mergers, and
3. limited liability.

Theorem (Existence)

When each agent’s preferences are substitutable, a competitive equilibrium exists.

Theorem (Solution Concept Equivalence)

When each agent’s preferences are substitutable, an outcome is

\[ CE \iff \text{Stable} \iff \text{Chain stable}. \]
The Setting: Trades and Contracts

Finite set of agents $I$.

Finite set of bilateral trades $\Omega$:
Each trade $\omega \in \Omega$ has a seller $s(\omega) \in I$ and a buyer $b(\omega) \in I$.

$\Psi_i \equiv \{ \psi \in \Psi : b(\psi) = i \}$.

$\Psi_i \equiv \{ \psi \in \Psi : s(\psi) = i \}$.

$\Psi \equiv \Psi \rightarrow i \cup \Psi \rightarrow i$.

An arrangement is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$.

Set of contracts $X \equiv \Omega \times \mathbb{R}$ each contract $x \in X$ is a pair $(\omega, p_\omega)$.

$\tau(Y) \subseteq \Omega$ set of trades in contract set $Y \subseteq X$.

A (feasible) outcome is a set of contracts $A \subseteq X$ which uniquely prices each trade in $A$.
The Setting: Trades and Contracts

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- Finite set of *agents* \( I \).
- Finite set of bilateral *trades* \( \Omega \):
  - Each trade \( \omega \in \Omega \) has a seller \( s(\omega) \in I \) and a buyer \( b(\omega) \in I \).
  - \( \Psi_{\rightarrow i} \equiv \{ \psi \in \Psi : b(\psi) = i \} \).
  - \( \Psi_{i\rightarrow} \equiv \{ \psi \in \Psi : s(\psi) = i \} \).
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- A (feasible) outcome is a set of contracts \( A \subseteq X \) which uniquely prices each trade in \( A \).
The Setting: Demand

- Each agent $i$ has quasilinear utility over arrangements:

$$U_i([\Psi; p]) = u_i(\Psi_i) - \sum_{\psi \in \Psi \rightarrow i} p_\psi + \sum_{\psi \in \Psi \leftarrow i} p_\psi.$$

- $U_i$ extends naturally to (feasible) outcomes.
Each agent $i$ has quasilinear utility over arrangements:

$$U_i([\Psi; p]) = u_i(\Psi_i) - \sum_{\psi \in \Psi \leftrightarrow i} p_\psi + \sum_{\psi \in \Psi_i \rightarrow} p_\psi.$$ 

$U_i$ extends naturally to (feasible) outcomes.

For any price vector $p \in \mathbb{R}^\Omega$, the demand of $i$ is

$$D_i(p) = \arg \max_{\psi \subseteq \Omega} U_i([\psi; p]).$$

For any set of contracts $Y \subseteq X$, the choice of $i$ is

$$C_i(Y) = \arg \max_{Z \subseteq Y} U_i(Z).$$
Assumptions on Preferences

1. \( u_i(\emptyset) \in \mathbb{R} \cup \{-\infty\} \).

2. Full substitutability...
Assumptions on Preferences

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2. \( u_i(\emptyset) \in \mathbb{R} \).
3. **Full substitutability**...
Choice-Language Full Substitutability

Definition

The preferences of agent $i$ are **fully substitutable in choice language** if for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$,

1. if $Y_i \rightarrow = Z_i \rightarrow$, and $Y_i \rightarrow \subseteq Z_i \rightarrow$,
   then for $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$,
   we have $(Y_i \rightarrow \setminus Y_i^*) \subseteq (Z_i \rightarrow \setminus Z_i^*)$ and $Y_i^* \subseteq Z_i^*$;

2. if $Y_i \rightarrow = Z_i \rightarrow$, and $Y_i \rightarrow \subseteq Z_i \rightarrow$,
   then for $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$,
   we have $(Y_i \rightarrow \setminus Y_i^*) \subseteq (Z_i \rightarrow \setminus Z_i^*)$ and $Y_i^* \subseteq Z_i^*$. 
Demand-Language Full Substitutability

**Definition**

The preferences of agent $i$ are **fully substitutable** in demand language if for all $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

1. if $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$,
   then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

   $$\left\{ \omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega \right\} \subseteq \Psi_{\rightarrow i}, \quad \Psi_{i\rightarrow} \subseteq \Psi'_{i\rightarrow};$$

2. if $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$,
   then for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have

   $$\left\{ \omega \in \Psi_{i\rightarrow} : p_\omega = p'_\omega \right\} \subseteq \Psi_{i\rightarrow}, \quad \Psi_{\rightarrow i} \subseteq \Psi'_{\rightarrow i}.$$
Indicator-Language Full Substitutability

\[ e_{i,\omega}(\Psi) = \begin{cases} 
1 & \omega \in \Psi_{\rightarrow i} \\
-1 & \omega \in \Psi_{\leftarrow i} \\
0 & \text{otherwise}
\end{cases} \]

**Definition**

The preferences of agent \( i \) are **fully substitutable** in indicator language if for all price vectors \( p, p' \in \mathbb{R}^\Omega \) such that \( |D_i(p)| = |D_i(p')| = 1 \) and \( p \leq p' \), for \( \Psi \in D_i(p) \) and \( \Psi' \in D_i(p') \), we have

\[ e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi') \]

for each \( \omega \in \Omega_i \) such that \( p_\omega = p'_\omega \).
Equivalence

Theorem

Choice-Language Full Substitutability $\iff$ Demand-Language Full Substitutability $\iff$ Indicator-Language Full Substitutability.

Allowing for definitions that look at price vectors for which the choice function is not single-valued is straightforward.
Equivalence

**Theorem**

Choice-Language Full Substitutability $\iff$ Demand-Language Full Substitutability $\iff$ Indicator-Language Full Substitutability.

Allowing for definitions that look at price vectors for which the choice function is not single-valued is straightforward.
Submodular Indirect Utility Functions

Definition

The \textbf{indirect utility function} of agent $i$, 

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} \{ U_i([\Psi; p]) \},$$

is \textbf{submodular} if, for all price vectors $p, \bar{p} \in \mathbb{R}^\Omega$, we have that

$$V_i(p \land \bar{p}) + V_i(p \lor \bar{p}) \leq V_i(p) + V_i(\bar{p}).$$
Submodular Indirect Utility Functions

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**Theorem**

A utility function is fully substitutable if and only if its associated indirect utility function is submodular.
Let

$$\sigma_i(\Psi) \equiv \{\sigma(\omega) : \omega \in \Psi_{\rightarrow i}\} \cup \{\sigma(\omega) : \omega \in \Omega_i \setminus \Psi_{\rightarrow i}\}.$$
Let

$$\sigma_i(\Psi) \equiv \{ \sigma(\omega) : \omega \in \Psi_{\rightarrow i} \} \cup \{ \sigma(\omega) : \omega \in \Omega_i \setminus \Psi_{\rightarrow i} \}.$$ 

**Definition**

A utility function is object-language fully substitutable if the induced utility function over objects is grossly substitutable (in the sense of Kelso & Crawford (1982)).
Let

\[ o_i(\Psi) \equiv \{ o(\omega) : \omega \in \Psi_i \} \cup \{ o(\omega) : \omega \in \Omega_i \setminus \Psi_i \}. \]

**Definition**

A utility function is object-language fully substitutable if the induced utility function over objects is grossly substitutable (in the sense of Kelso & Crawford (1982)).

**Theorem**

A utility function is fully substitutable if and only if it is object-language fully substitutable.
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<thead>
<tr>
<th>The Single Improvement Property</th>
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</thead>
</table>

**Classic Idea**

When $i$ does not have an optimal bundle, he can make himself better off by adding a single item, dropping a single item, or both.
The Single Improvement Property

**Definition**

The preferences of agent \( i \) have the **single improvement property** if whenever \( i \) holds a suboptimal bundle of trades \( \Psi \), he can be made be better off by

1. obtaining one object not currently held (either by making a new purchase, i.e., adding a trade in \( \Omega_{i \rightarrow} \setminus \Psi \), or by canceling a sale, i.e., removing a trade in \( \Psi_{i \rightarrow} \)),
2. relinquishing one object currently held (either by canceling a purchase, i.e., removing a trade in \( \Psi_{i \rightarrow} \), or by making a new sale, i.e., adding a trade in \( \Omega_{i \rightarrow} \setminus \Psi \)), or
3. both obtaining one object not currently held and relinquishing one object currently held.

Fully substitutable choice functions satisfy the single improvement property (Gul & Stacchetti (1999) + object-language full substitutability).
Suppose an agent $i$ is endowed with the right to execute trades in the set $\Phi \subseteq \Omega_i$ at prices $p_\Phi$. Let

$$\hat{u}_i^{(\Phi, p_\Phi)}(\Psi) \equiv \max_{\Xi \subseteq \Phi} \left\{ u_i(\Psi \cup \Xi) + \sum_{\xi \in \Xi \rightarrow i} p_\xi - \sum_{\xi \in \Xi \rightarrow i} p_\xi \right\}$$

be the induced valuation over trades in $\Omega \setminus \Phi$. 

Theorem

If the preferences of agent $i$ are fully substitutable, then the preferences induced by $\hat{u}_i^{(\Phi, p_\Phi)}(\Psi)$ are, as well.
Suppose an agent $i$ is endowed with the right to execute trades in the set $\Phi \subseteq \Omega_i$ at prices $p_\Phi$. Let

$$\hat{u}_i^{(\Phi, p_\Phi)}(\Psi) \equiv \max_{\Xi \subseteq \Phi} \left\{ u_i(\Psi \cup \Xi) + \sum_{\xi \in \Xi \to i} p_\xi - \sum_{\xi \in \Xi \to i} p_\xi \right\}$$

be the induced valuation over trades in $\Omega \setminus \Phi$.

**Theorem**

If the preferences of agent $i$ are fully substitutable, then the preferences induced by $\hat{u}_i^{(\Phi, p_\Phi)}$ are, as well.
The indirect utility function for $\hat{u}_i^{(\Phi,p_\Phi)}$ is given by

$$\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\setminus\Phi}) \equiv \max_{\psi \subseteq \Omega \setminus \Phi} \left\{ \max_{\Xi \subseteq \Phi} \left\{ \left( u_i(\psi \cup \Xi) + \sum_{\xi \in \Xi \rightarrow i} p_\xi - \sum_{\xi \in \Xi \rightarrow i} p_\xi \right) \right\} + \sum_{\psi \in \psi_i} p_\psi - \sum_{\psi \in \psi_i} p_\psi \right\}$$

$$= \max_{\psi \subseteq \Omega \setminus \Phi} \left\{ \max_{\Xi \subseteq \Phi} \left\{ u_i(\psi \cup \Xi) + \sum_{\lambda \in \Xi \rightarrow \psi_i} p_\lambda - \sum_{\lambda \in \Xi \rightarrow \psi_i} p_\lambda \right\} \right\}$$

$$= \max_{\Lambda \subseteq \Omega} \left\{ u_i(\Lambda) + \sum_{\lambda \in \Lambda \rightarrow i} p_\lambda - \sum_{\lambda \in \Lambda \rightarrow i} p_\lambda \right\}$$

$$= V_i(p_{\Omega \setminus \Phi}, p_\Phi).$$

$V_i(p)$ is submodular over $\mathbb{R}^\Omega \Rightarrow \hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega \setminus \Phi})$ is submodular over $\mathbb{R}^{\Omega \setminus \Phi}$. 
Suppose an agent $i$ is obliged to execute trades in the set $\Phi \subseteq \Omega_i$ at prices $p_\Phi$. Let

$$\tilde{u}_i^{(\Phi, p_\Phi)}(\Psi) \equiv u_i(\Psi \cup \Phi) + \sum_{\varphi \in \Phi \rightarrow i} p_{\varphi} - \sum_{\varphi \in \Phi \rightarrow i} p_{\varphi}$$

be the induced valuation over trades in $\Omega \setminus \Phi$. 
Suppose an agent $i$ is obliged to execute trades in the set $\Phi \subseteq \Omega_i$ at prices $p_\Phi$. Let

$$\tilde{u}_i^{(\Phi, p_\Phi)}(\Psi) \equiv u_i(\Psi \cup \Phi) + \sum_{\varphi \in \Phi \rightarrow i} p_\varphi - \sum_{\varphi \in \Phi \rightarrow i} p_\varphi$$

be the induced valuation over trades in $\Omega \setminus \Phi$.

**Theorem**

*If the preferences of agent $i$ are fully substitutable, then the preferences induced by $\tilde{u}_i^{(\Phi, p_\Phi)}$ are as well (so long as $u_i(\Phi) \neq -\infty$).*
For $J \subseteq I$, we denote $\Omega^J \equiv \{ \omega \in \Omega : \{ b(\omega), s(\omega) \} \subseteq J \}$. The **convolution** of the valuation functions $\{ u_j \}_{j \in J}$ is

$$u_J(\Psi) \equiv \max_{\Phi \subseteq \Omega^J} \left\{ \sum_{j \in J} u_j(\Psi \cup \Phi) \right\},$$

defined over $\Psi \subseteq \Omega \setminus \Omega^J$. 

**Theorem** If the preferences of each $j \in J$ are fully substitutable, then the preferences induced by the convolution $u_J$ are, as well.
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**Theorem**

*If the preferences of each $j \in J$ are fully substitutable, then the preferences induced by the convolution $u_J$ are, as well.*
Transformations: Limited Liability

Take an arbitrary set of trades $\Phi \subseteq \Omega_i$, and for every trade $\varphi \in \Phi$, pick $\Pi_\varphi \in \mathbb{R}$—the penalty for reneging on trade $\varphi$. Define the modified valuation function $\hat{u}_i$ as

$$\hat{u}_i(\Psi) \equiv \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \setminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}.$$
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$$\hat{u}_i(\Psi) \equiv \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \setminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}.$$

**Theorem**

For any $\Phi \subseteq \Omega_i$ and $\Pi_\Phi \in \mathbb{R}^\Phi$, if the $u^i$ is fully substitutable, then $\hat{u}_i$ is fully substitutable.
Implications of Full Substitutability

**Theorem (Hatfield et al. (2013, 2020))**

In arbitrary trading networks with fully substitutable preferences,
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Theorem (Hatfield et al. (2013, 2020))

In arbitrary trading networks with fully substitutable preferences,

1. A competitive equilibrium exists (and is efficient), and
Theorem (Hatfield et al. (2013, 2020))

In arbitrary trading networks with fully substitutable preferences,

1. A competitive equilibrium exists (and is efficient), and
2. An outcome is a competitive equilibrium if and only if it is stable if and only if it is chain stable.
Implications of Full Substitutability

Theorem (Hatfield et al. (2013, 2020))

In arbitrary trading networks with fully substitutable preferences,

1. A competitive equilibrium exists (and is efficient), and
2. An outcome is a competitive equilibrium $\iff$ it is stable $\iff$ it is chain stable.

It is also computationally “easy” to find competitive equilibria in this setting (Candogan et al., 2018)
Open Questions

1. Characterizing substitutability—Can we "generate" all of the fully substitutable choice functions from a set of simple choice functions and operations? (Tran, 2020; Paes Leme, 2020).

2. Do these operations have economic meaning?

3. Can we characterize substitutability in settings without transfers? Much less seems to be known here.

4. Without transfers, all of our nice results require that the network structure be acyclic—why?
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   - Much less seems to be known here.
Open Questions

1. Characterizing substitutability—Can we “generate” all of the fully substitutable choice functions from a set of simple choice functions and operations? (Tran, 2020; Paes Leme, 2020).
   - Do these operations have economic meaning?

2. Can we characterize substitutability in settings without transfers?
   - Much less seems to be known here.

3. Without transfers, all of our nice results require that the network structure be acyclic—why?